VERIFICATION OF THE GOVERNING EQUATIONS FOR THE NONLINEAR DEFORMATION OF MATERIALS WITH DIFFERENT STRENGTHS IN TENSION AND COMPRESSION

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1. The construction of governing equations for isotropic media with dissimilar properties in tension and compression involves determination of the relationship between the components of the tensors h_{ij} and σ_{ij} characterizing the kinematic and force factors. The most general relation between these tensors was obtained in [1]. The governing equations proposed in [2-9] can be considered different forms of the relations obtained by V. V. Novozhilov.

We will examine the construction of physical equations containing the first $I_1 = \sigma_{ij}\delta_{ij}$ and second $I_2 = (\sigma_{ij}\sigma_{ij} - I_1^2)/2$ invariants of the tensor σ_{ij} (δ_{ij} is the Kronecker symbol). To do this, we assume the existence of the potential

$$W = \sigma_e^2/2, \tag{1.1}$$

where $\sigma_e \geqslant 0$;

$$\sigma_e = \sigma + \sigma_0,\ \sigma = BI_1,\ \sigma_0^2 = AI_1^2 + 4CI_2 \tag{1.2}$$

(A, B, and C are constants). Then the components of the tensor h_{ij} are determined by the law $h_{ij} = \lambda\partial W/\partial\sigma_{ij}$, where λ is a scalar multiplier. Then using Eqs. (1.1) and (1.2) and the relation

$$\partial W/\partial\sigma = \partial W/\partial\sigma_0 = \sigma_e,\ \partial\sigma/\partial\sigma_{ij} = B\delta_{ij},$$
$$\partial\sigma_0/\partial\sigma_{ij} = [(A - 2C)I_1\delta_{ij} + 2C\sigma_{ij}]/\sigma_0,$$

we have

$$h_{ij} = \lambda\sigma_e\left[\frac{(A-2C)\,I_1\delta_{ij} + 2C\sigma_{ij}}{\sigma_0} + B\delta_{ij}\right]. \tag{1.3}$$

To find the multiplier λ, we form the mixed invariant $L = \sigma_{ij}h_{ij}$ and from (1.3) we obtain $L = \lambda\sigma_e^2$.

We then arrive at the linear-tensor relations

$$h_{ij} = \frac{L}{\sigma_e}\left[\frac{(A-2C)\,I_1\delta_{ij} + 2C\sigma_{ij}}{\sigma_0} + B\delta_{ij}\right], \tag{1.4}$$

determining the behavior of materials with different strength in tension and compression.

2. Let us examine elastic deformation. In this case, $h_{ij} \equiv e_{ij}$ and σ_{ij} in (1.4) are components of the tensors of the elastic strains and stresses, respectively; $\lambda = 1$, so that $L = \sigma_e^2$. Physical equations (1.4) contain three types of constants: $A \equiv A^0$, $B \equiv B^0$, $C \equiv C^0$, which can be found, for example, on the basis of standard uniaxial tests in tension (+) and compression (−) with determination of the elastic moduli E_+ and E_- and the Poisson's ratio ν_+:

$$A^0 = [(E_+)^{-1/2} + (E_-)^{-1/2}]^2/4,$$
$$B^0 = [(E_+)^{-1/2} - (E_-)^{-1/2}]/2,\ C^0 = (A^0 + B^0\sqrt{A^0})(1 + \nu_+)/2. \tag{2.1}$$

The equality $E_+ = E_-$ is valid for materials with the same properties in tension and compression. Then in Eq. (2.1) $B^0 = 0$, while in (1.2) $\sigma = 0$. Here, governing equations (1.4) reduce to Hooke's law

$$e_{ij} = (A^0 - 2C^0)I_1\delta_{ij} + 2C^0\sigma_{ij}. \tag{2.2}$$

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3. We will obtain the relation between the components of the tensors of the rates of plastic strain $\dot{p}_{ij}$ and the stresses σ_{ij} in an elastoplastic body. The dot above the symbol denotes a derivative with respect to the loading parameter t. We will use Eqs. (1.4), where $h_{ij} \equiv \dot{p}_{ij}$; $A \equiv A_0$, $B \equiv B_0$, $C \equiv C_0$ are scalar parameters; L is the specific density of the energy dissipated during plastic deformation; σ_e is the equivalent stress.

As the strain-hardening measure we take $q = \int \frac{L}{\sigma_e} dt$. We write the plastic strain condition in the form

$$\sigma_e = \varphi(q). \tag{3.1}$$

Elastic deformation occurs at $\sigma_e = \varphi(q)$, so that L = 0. If condition (3.1) is satisfied and additionally $\dot{\sigma}_e > 0$, then loading takes place and L > 0. If Eq. (3.1) is valid and the relation $\dot{\sigma}_e \leq 0$ is satisfied, then L = 0, and either unloading or neutral loading takes place.

We now examine the case of simple loading when the components of the stress tensor increase in proportion to the parameter $t \in [0, 1]$: $\sigma_{ij} = t\sigma^*_{ij}$. The asterisk denotes values at the end of loading. Performing the operation of integration, we change Eqs. (1.4) to the form

$$p^*_{ij} = \left[\frac{(A_0 - 2C_0) I^*_1 \delta_{ij} + 2C_0 \sigma^*_{ij}}{\sigma^*_0} + B_0 \delta_{ij}\right] \int\limits_0^1 \frac{L}{\sigma_e} dt.$$

Using the definition of the strain-hardening measure, we obtain $q^* = \int\limits_0^1 \frac{L}{\sigma_e} dt$ and then, on the basis of (3.1), we obtain $q^* = v(\sigma^*_e)$. We finally arrive at the following physical relations (with the asterisk omitted) for the strain variant of the theory:

$$p_{ij} = v(\sigma_e) \left[\frac{(A_0 - 2C_0) I_1 \delta_{ij} + 2C_0 \sigma_{ij}}{\sigma_0} + B_0 \delta_{ij}\right]. \tag{3.2}$$

The function $v(\sigma_e)$ can be determined either from a tabulation or by analytical means. The simplest methods of assigning this function are as a power relation σ_e^n, a hyperbolic sine law $\sinh(\sigma_e/a)$, and an exponential relation $\exp(\sigma_e/b)$ (n, a, and b are material constants). Although the natural condition v(0) = 0 is not satisfied for the last representation, this function is often used to approximate stress–strain curves.

Let us discuss the method of determining the constants in derived equations (3.2) on the basis of data from standard tests. Let the following relation hold for the uniaxial tension of specimens

$$p_{11} = D\sigma^n_{11}. \tag{3.3}$$

We also assume that tests were conducted in torsion with the shear stresses σ_{12} and that we experimentally found the relationship between the principal stresss $\sigma_{11} = \sigma_{12}$ and $\sigma_{33} = -\sigma_{12}$ ($\sigma_{22} = 0$) and the corresponding principal strains, i.e., the strains in the principal directions

$$p_{11} = T_+ \sigma^n_{11}, \quad p_{33} = -T_- |\sigma_{33}|^n. \tag{3.4}$$

Then, knowing the material constants D, T_+, T_-, and n and comparing Eqs. (3.3) and (3.4) with the analogous relations which follow from (3.2), we determine the following three scalar parameters if we put $v(\sigma_e) = \sigma_e^n$:

$$C_0 = (T_+ + T_-)^{\frac{2}{n+1}}/4, \quad B_0 = (T_+ - T_-)\Big/\left(2^{n+1} C_0^{\frac{n}{2}}\right), \quad A_0 = \left(D^{\frac{1}{n+1}} - B_0\right)^2. \tag{3.5}$$

For materials for which $T_+ = T_- = 3^{\frac{n+1}{2}} D/2$, we find from (3.5) that $B_0 = 0$, $4C_0 = 3A_0$. Then it follows from (1.2) that $\sigma = 0$, and the equivalent stress coincides to within the constant with the stress intensity σ_i, i.e., $\sigma_e = \sqrt{A_0}\sigma_i$. In this case, physical relations (3.2) take the form

$$p_{ij} = \frac{3}{2} (\sqrt{A_0})^{n+1} \sigma_i^{n-1} \left(\sigma_{ij} - \frac{1}{3} I_1 {}_{ij}\right),$$

familiar from the classical theory of plasticity.

To verify governing equations (3.2), we will compare the theoretical results with experimental results for a complex stress state. We will therefore examine the elastoplastic deformation of gray iron of a composition close to the grade SCh 15-32 [6]. The tests were conducted in uniaxial tension and pure torsion (with measurement of the principal strains ε_{11}, ε_{33} of the specimens of this material). The tests established that the elastoplastic properties were isotropic, the elastic strains are independent of the type of loading, and the material has different resistances to plastic deformation. Figure 1 shows stress–strain curves of the gray iron: 1 is for uniaxial tension and 2 and 3 are for torsion (3 corresponds to the direction of the maximum principal stress σ_{11}, 2 corresponds to the direction of the minimum principal stress [σ_{33}]). Analysis of the stress–strain curves leads to the following relations: for uniaxial tension

$$\varepsilon_{11} = e_{11} + p_{11},\ e_{11} = M\sigma_{11},\ p_{11} = D_{11}\sigma_{11}^{n},$$

for torsion

$$\varepsilon_{11} = e_{11} + p_{11},\ e_{11} = Q\sigma_{11},\ p_{11} = T_{+}\sigma_{11}^{n},$$
$$\varepsilon_{33} = e_{33} + p_{33},\ e_{33} = -Q|\sigma_{33}|,\ p_{33} = -T_{-}|\sigma_{33}|^{n}.$$

The values of the material constants: $n = 4.5$, $M = 10^{-4}$ mm^2/kg, $Q = 1.40\cdot10^{-4}$, $D = 1.32\cdot10^{-8}$ (mm^2/kg)^n, $T_+ = 4.06\cdot10^{-8}$, $T_- = 1.59\cdot10^{-8}$. We then use these values to find the parameters of Hooke's law (2.2) and physical equations (3.2), and for any stress state we calculate the strains $\varepsilon_{ij} = e_{ij} + p_{ij}$.

Let us discuss the results of tests of thin-walled tubular specimens of the same material loaded by internal pressure and an axial force [6]. Solid lines 1 and 2 in Fig. 2 show the experimental relations $\varepsilon_{11} - \sigma_{11}$ and $\varepsilon_{22} - \sigma_{22}$ with the ratio $\sigma_{11}/\sigma_{22} = 2.3$. For comparison, the dashed lines show the analogous theoretical results. It can be seen from Fig. 2 that the agreement between the theoretical and experimental results can be considered satisfactory.

4. We will obtain the governing equations for the creep of strain-hardening materials having different strengths in tension and compression. To do this, we use Eqs. (1.4), assuming that $h_{ij} \equiv \dot{z}_{ij}$ and that σ_{ij} are components of the tensors of the rates of creep strain and the stresses, respectively; $A \equiv A^*$, $B \equiv B^*$, $C \equiv C^*$ are constants; L is the specific density of the energy dissipated during creep; σ_e is the equivalent stress. The dot above the symbol denotes differentiation with respect to the time t. To describe the strain-hardening of materials, we introduce the parameter q, characterized by the kinetic equation,

$$dq/dt = R. \tag{4.1}$$

The right side of (4.1) can be defined in several ways. For example, we can take $R = 1$, $R = L$, or $R = L/\sigma_e$.

We assume that the energy density is a function of the equivalent stress and the structural parameter q

$$L = \psi(\sigma_e,\ q).$$

We take $L = \sigma_e v(\sigma_e)\chi(q)$. Then physical equations (1.4) are written as follows

$$\dot{z}_{ij} = v(\sigma_e)\,\chi(q)\left[\frac{(A^* - 2C^*)\,I_1\delta_{ij} + 2C^*\sigma_{ij}}{\sigma_0} + B^*\delta_{ij}\right]. \tag{4.2}$$

The function $v(\sigma_e)$ can be taken in one of the following forms: a power relation σ_e^n, a hyperbolic sine law $\sinh(\sigma_e/a)$ with known stipulations, or an exponential relation $\exp(\sigma_e/b)$. The representations q^m or $\exp(q/c)$ are possible for the function $\chi(q)$. Here, n, a, b, m, and c are constants. The correct selection of the functions $v(\sigma_e)$ and $\chi(q)$ is related to questions regarding the best approximation of the creep curves and is done on the basis of data from the standard tests. For non-strain-hardening materials, we should take $\chi = 1$. In this case, governing equations (4.2) become the relations which were proposed and verified experimentally in [9].

Let us examine the method of determining the parameters which enter into derived equations (4.2). To do this, we will use a series of independent standard tests involving the tension, compression, and torsion of specimens of the investigated strain-hardening material under creep conditions with constant stresses. Let the relationship between the strain rate

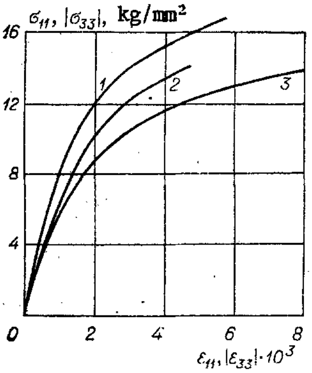

Fig. 1

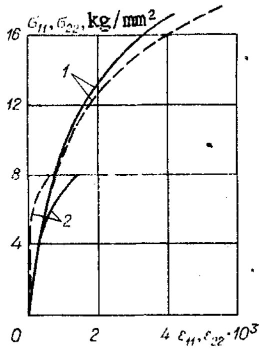

Fig. 2

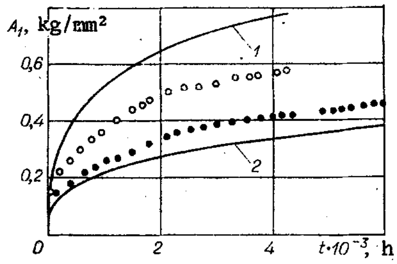

Fig. 3

$\dot{Z}$ and the stress σ_* be established from analysis of the creep curves, i.e., for example, $\dot{Z} = K_+ m t^{m-1} \sigma_*^n$ in uniaxial tension and $\dot{Z} = -K_- m t^{m-1} |\sigma_*|^n$ ($K_- > 0$) in uniaxial compression. We assume that a similar relation holds between the rate of shear creep $\dot{\gamma}$ and the shear stress τ in torsion: $\dot{\gamma} = K m t^{m-1} \tau_*^n$. Here, K_+, K_-, K, m, n are material constants.

Then we can take $v(\sigma_e) = \sigma_e^n$ and $\chi(q) = m q^{m-1}$ in physical relations (4.2) and $R = 1$ in (4.1), i.e., $q = t$. Then writing Eqs. (4.2) for each of the three above-examined cases of a unidimensional stress state and comparing them with the analogous relations presented earlier, we find the constants

$$A^* = [K_+^{1/(n+1)} + K_-^{1/(n+1)}]^2/4,$$
$$B^* = [K_+^{1/(n+1)} - K_-^{1/(n+1)}]/2, \quad C^* = K^{2/(n+1)}/4. \tag{4.3}$$

If we establish from the standard tests that $K_+ = K_-$, $K = 3^{\frac{n+1}{2}} K_+$, then from (4.3) we find

$$A^* = K_+^{2/(n+1)}, \quad B^* = 0, \quad C^* = 3K_+^{2/(n+1)}/4.$$

In this case, $\sigma = 0$, $\sigma_0 = \sqrt{A^*}\sigma_i$, σ_i is the stress intensity. Then governing equations (4.2) become the well-known equations

$$\dot{Z}_{ij} = \frac{3}{2} m (\sqrt{A^*})^{n+1} \sigma_i^{n-1} q^{m-1} \left(\sigma_{ij} - \frac{1}{3} I_1 \delta_{ij}\right),$$

describing the creep of materials with the same resistance to deformation under different types of loading.

For example, analysis of data from creep tests of aluminum alloy VT-9 at 400°C [7] in tension, compression, and torsion leads to the values $n = 5.91$, $m = 0.265$, $K_+ = 1.45 \cdot 10^{-14}$ $(\text{kg/mm}^2)^n \text{h}^{-m}$, $K_- = 5.10 \cdot 10^{-15}$, $K = 3.20 \cdot 10^{-13}$. It is evident that this strain-hardening material manifests a substantial difference in strength in tension and compression.

We can then use Eqs. (4.3) to find the parameters A^*, B^*, and C^* and use Eqs. (4.2) to describe the creep of this titanium alloy in a complex stress state. To do this, we compare the theoretical results with data from the experiments in [7] conducted on thin-walled tubes loaded by torsion and an axial (tensile or compressive) force. Figure 3 shows the change in specific work $A_1 = \sigma_{11} Z_{11} + 2\sigma_{12} Z_{12}$ with time. The test data is denoted by circles. The clear circles pertain to tension with torsion ($\sigma_{11} = 56$ kg/mm², $\sigma_{12} = 26.5$), while the dark circles correspond to compression with torsion ($\sigma_{11} = -56$, $\sigma_{12} = 26.5$). Lines 1 and 2 are the analogous theoretical results. Considering the appreciable difference in the strengths of the material and the natural scatter of creep data, the agreement between the theoretical and experimental results can be adjudged satisfactory.

Thus, the proposed physical relations can be used to describe the elasticity, plasticity, and creep of isotropic materials with different strengths in tension and compression.

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PROBLEM OF THE SYNTHESIS OF A COMPOSITE MATERIAL OF UNIDIMENSIONAL STRUCTURE WITH ASSIGNED CHARACTERISTICS

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A significant amount of attention is now being given to the development of composite materials with assigned properties. Here we present a solution of this problem in regard to the thermophysical and stiffness characteristics of composites with a unidimensional structure (with the condition that the components have the same Poisson's ratios).

By a composite material with a unidimensional structure we mean an inhomogeneous material with thermophysical and mechanical characteristics which are a function of a single space variable, such as x_1. Composites constitute a special case of such materials. The characteristics of a composite composed of a large number of small components are rapidly oscillating functions with a characteristic magnitude of oscillation $\varepsilon \ll 1$ (in the case of laminated composites, ε is the characteristic thickness of the layers). As was shown in [1-5], at $\varepsilon \to 0$ an inhomogeneous composite with a periodic structure can be regarded as a homogeneous material with so-called averaged [1-5] thermophysical and mechanical characteristics which at $\varepsilon \ll 1$ are close to the thermomechanical behavior of the original material [1-6]. The averaged characteristics, describing the material from the macroscopic viewpoint, are determined by its its local (microscopic) characteristics. The question of determining averaged characteristics of composites from their local characteristics has been fully resolved by now [1-7]. Here we examine the inverse problem: through which averaged characteristics and in what manner can we impart a unidimensional structure to composites by controlling their local characteristics? The solution is obtained on the basis of the methods used in [8, 9] in regard to thermophysical and stiffness characteristics.

Let the composite material we are studying be locally isotropic and inhomogeneous, with a periodic structure. The characteristic size of the period $\varepsilon \ll 1$. We apply the following restriction to the types of composites for which our findings are valid: the materials used in the composite must have the same (or similar) Poisson's ratios. This condition is met, for example, by a composite based on metals ($\nu \approx 1/3$) or polymers ($\nu \approx 0.4$). The material characteristics of the composites being examined: $c(x_1/\varepsilon)$, $a(x_1/\varepsilon)$ are the local heat capacity and thermal conductivity; $E(x_1/\varepsilon)$, $A(x_1/\varepsilon)$ are the local Young's modulus and coefficient of linear expansion [the period of the functions $c(t)$, $a(t)$ $E(t)$, $A(t)$ is equal to unity]. At $\varepsilon \to 0$, the solutions of the heat-conduction and strain problems for the composite approach [in the norm of the space $L_2(Q)$] the solutions of the same problem for a homogeneous anisotropic material with averaged characteristics:

heat capacity [2, 7]

$$\hat{c} = \langle c \rangle, \tag{1}$$

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